

QED Renormalization Given in A Mass-Dependent Subtraction and The Renormalization Group Approach

Jun-Chen Su, Xue-Xi Yi and Ying-Hui Cao

Center for Theoretical Physics, Department of Physics,
Jilin University, Changchun 130023,
People's Republic of China

Abstract

The QED renormalization is restudied by using a mass-dependent subtraction which is performed at a time-like renormalization point. The subtraction exactly respects necessary physical and mathematical requirements such as the gauge symmetry, the Lorentz- invariance and the mathematical convergence. Therefore, the renormalized results derived in the subtraction scheme are faithful and have no ambiguity. Especially, it is proved that the solution of the renormalization group equation satisfied by a renormalized wave function, propagator or vertex can be fixed by applying the renormalization boundary condition and, thus, an exact S-matrix element can be expressed in the form as written in the tree diagram approximation provided that the coupling constant and the fermion mass are replaced by their effective ones. In the one-loop approximation, the effective coupling constant and the effective fermion mass obtained by solving their renormalization group equations are given in rigorous and explicit expressions which are suitable in the whole range of distance and exhibit physically reasonable asymptotic behaviors.

PACS: 11.10Gh, 12.20Ds

Keywords: QED renormalization, mass-dependent subtraction, time-like renormalization point, removal of ambiguity, exact one-loop results.

1. Introduction

In the renormalization of quantum field theories, to extract finite physical results from higher order perturbative calculations, a certain subtraction scheme is necessary to be used so as to remove the divergences occurring in the calculations. There are various subtraction schemes in the literature, such as the minimal subtraction (MS)¹, the modified minimal subtraction (\overline{MS})², the momentum space subtraction (MOM)³, the on-mass-shell subtraction (OS)^{4,5}, the off-mass shell subtraction (OMS)⁶ and etc.. However, there exists a serious ambiguity problem^{3,6,7} that different subtraction schemes in general give different physical predictions, conflicting the fact that the physical observables are independent of the subtraction schemes. Ordinarily, it is argued that the ambiguity appears only in finite order perturbative calculations, while the exact result given by the whole perturbation series is scheme- independent. Though only a finite order perturbative calculation is able to be done in practice, one still expects to get unambiguous results from such a calculation. To solve the ambiguity problem, several prescriptions, such as the minimal sensitivity principle⁷, the effective charge method⁸ and some others⁹ , were proposed in the past. By the principle of minimal sensitivity, an additional condition has to be introduced and imposed on the result calculated in a finite order perturbative approximation so as to obtain an optimum approximant which is least sensitive to variations in the unphysical parameters. In the effective charge method, the use of a coupling constant is abandoned. Instead, an effective charge is associated with each physical quantity and used to determine the Gell-Mann-Low function in the renormalization group equation (RGE). In this way, a renormalization-scheme-invariant result can be found from the RGE. In Ref.(9), the authors developed a new perturbation approach to renormalizable field theories. This approach is based on the observation that if the perturbation series of a quantity R is not directly computable due to that the expansion coefficients are infinite, the unambiguous result of the quantity can be found by solving the differential equation $Q \frac{dR(Q)}{dQ} = F(R(Q))$ where $F(R)$ is well-defined and can be expanded as a series of R . Particularly, the finite coefficients of the expansion of $F(R)$ are renormalization-scheme-independent and in one-to-one correspondence with the ones in the ordinary perturbation series of the R . The prescriptions mentioned above are somehow different from the conventional perturbation theory in which the coupling constant is commonly

chosen to be the expansion parameter for a perturbation series.

In this paper, we wish to deal with the ambiguity problem from a different angle. It will be shown that the ambiguity problem can be directly tackled in the conventional perturbation theory by the renormalization group method^{10–15}. The advantage of this method is that the anomalous dimension in a RGE is well-defined although it is computed from the renormalization constant which is divergent in its original definition. In comparison of the renormalization group method with the aforementioned approach proposed in Ref.(9), we see, both of them are much similar to one another in methodology. This suggests that the renormalization group method is also possible to yield the theoretical results which are free from the ambiguity. The possibility relies on how to choose a good subtraction scheme which gives rise to such renormalization constants that they lead to unique anomalous dimensions. The so-called good subtraction scheme means that it must respect necessary physical and mathematical requirements such as the gauge symmetry, the Lorentz invariance and the mathematical convergence principle. The necessity of these requirements is clear. Particularly, in the renormalization group approach to the renormalization problem, the renormalization constants obtained from a subtraction scheme not only serve to subtract the divergences, but also are directly used to derive physical results. Apparently, to guarantee the calculated results to be able to give faithful theoretical predictions, the subtraction procedure necessarily complies with the basic principles established well in physics and mathematics. Otherwise, the subtraction scheme should be discarded and thus the scheme-ambiguity will be reduced. Let us explain this viewpoint in some detail from the following aspects: (1) For a gauge field theory, as one knows, the gauge-invariance is embodied in the Ward identity. This identity is a fundamental constraint for the theory. Therefore, a subtraction scheme, if it is applicable, could not defy this identity. As will be demonstrated in latter sections, the Ward identity not only establishes exact relations between renormalization constants, but also determines the functional structure of renormalization constants; (2) The Lorentz-invariance is partly reflected in the energy-momentum conservation which holds at every vertex. Since the renormalization point in the renormalization constants will be eventually transformed to the momentum in the solutions of RGEs, obviously, in order to get correct functional relations of the solutions with the momentum, the energy-momentum conservation could not be violated by the subtraction of vertices; (3) Why the convergence

principle is required in the renormalization calculation? As we know, the renormalization constants are divergent in their original appearance. Such divergent quantities are not well-defined mathematically and hence are not directly calculable⁹because we are not allowed to apply any computational rule to do a meaningful or unambiguous calculation for this kind of quantities. The meaningful calculation can only be done for the regularized form of renormalization constants which are derived from corresponding regularized Feynman integrals in a subtraction scheme. The necessity of introducing the regularization procedure in the quantum field theory may clearly be seen from the mathematical viewpoint as illustrated in Appendix A. Therefore, in renormalization group calculations, the correct procedure of computing anomalous dimensions is starting from the regularized form of renormalization constants. The limit operation taken for the regularization parameter should be performed after completing the differentiation with respect to the renormalization point. Since the anomalous dimensions are convergent, the limit is meaningful and would give definite results. Obviously, the above procedure of computing the anomalous dimensions agrees with the convergence principle; (4) In comparison with the other subtraction schemes, the MOM scheme appears to be more suitable for renormalization group calculations. This is because this scheme naturally provides not only a renormalization point which is needed for the renormalization group calculation, but also a renormalization boundary condition for a renormalized quantity (a wave function, a vertex or a propagator), which will be used to fix the solution of the RGE for the renormalized quantity.

Based on the essential points of view stated above, it may be found that a renormalized quantity can be unambiguously determined by its RGE and thus a renormalized S-matrix element can be given in an unique form without any ambiguity. To illustrate this point, we limit ourselves in this paper to take the QED renormalization as an example to show how the ambiguity can be eliminated. As one knows, the QED renormalization has been extensively investigated by employing the OS, \overline{MS} and MOM schemes in the previous works^{4–6,10–26}. But, most of these studies are concentrated on the large momentum (short distance) behaviors of some quantities for the sake of simplicity of the calculation. In this paper, we restudy the QED renormalization with the following features: (1) The renormalization is performed in a mass-dependent scheme other than in the mass-independent scheme which was adopted in many previous works^{20–26}. The mass-dependent scheme obvi-

ously is more suitable for the case that the mass of a charged fermion can not be set to be zero; (2) The subtraction is carried out in such a MOM scheme that the renormalization point is mainly taken to be an arbitrary time-like momentum other than a space-like momentum as chosen in the conventional MOM scheme. The time-like MOM scheme actually is a generalized mass-shell scheme (GMS). The prominent advantage of the GMS scheme is that in this scheme the scale of renormalization point can naturally be connected with the scale of momenta and the results obtained can directly be converted to the corresponding ones given in the OS-scheme ; (3) The subtraction is implemented by fully respecting the necessary physical and mathematical principles mentioned before . Therefore, the results obtained are faithful and free of ambiguity; (4) The effective coupling constant and the effective fermion mass obtained in the one-loop approximation are given exact and explicit expressions which were never found in the literature. These expressions exhibit physically reasonable infrared and ultraviolet behaviors. We will pay main attention to the infrared (large distance) behavior because this behavior is more sensitive to identify whether a subtraction is suitable or not for the QED renormalization.

The rest of this paper is arranged as follows. In Sect.2, we sketch the RGE and its solution and show how a S-matrix element can be free of ambiguity. In Sect.3, we briefly discuss the Ward identity and give a derivation of the subtraction version of the fermion self-energy in the GMS scheme. In Sect.4, we derive an exact expression of the one-loop effective coupling constant and discuss its asymptotic property. In Sect.5, the same thing will be done for the effective fermion mass. The last section serves to make some comments and discussions. In Appendix A, we show a couple of mathematical examples to help understanding the regularization procedure used in the renormalization group calculations.

2.Solution to RGE and S-matrix element

Among different formulations of the RGE (see the review given in Ref.(14))^{10–15}, we like to employ the approach presented in Ref.(15). But, we work in a mass-dependent renormalization scheme, therefore, the anomalous dimension in the RGE depends not only on the coupling constant, but also on the fermion mass. Suppose F_R is a renormalized quantity. In the multiplicative

renormalization, it is related to the unrenormalized one F in such a way

$$F = Z_F F_R \quad (2. 1)$$

where Z_F is the renormalization constant of F . In GMS scheme, the Z_F and F_R are all functions of the renormalization point $\mu = \mu_0 e^t$ where μ_0 is a fixed renormalization point corresponding the zero value of the group parameter t . Differentiating Eq.(2.1) with respect to the μ and noticing that the F is independent of μ , we immediately obtain a RGE satisfied by the function F_R

$$\mu \frac{dF_R}{d\mu} + \gamma_F F_R = 0 \quad (2. 2)$$

where γ_F is the anomalous dimension defined by

$$\gamma_F = \mu \frac{d}{d\mu} \ln Z_F \quad (2. 3)$$

We first note here that because the renormalization constant is dimensionless, the anomalous dimension can only depends on the ratio $\sigma = \frac{m_R}{\mu}$, $\gamma_F = \gamma_F(g_R, \sigma)$, where m_R and g_R are the renormalized fermion mass and coupling constant respectively. Next, we note, Eq.(2.2) is suitable for a physical parameter (mass or coupling constant), a propagator, a vertex, a wave function or some other Green function. If the function F_R stands for a renormalized Green function, vertex or wave function, in general, it not only depends explicitly on the scale μ , but also on the renormalized coupling constant g_R , mass m_R and gauge parameter ξ_R which are all functions of μ , $F_R = F_R(p, g_R(\mu), m_R(\mu), \xi_R(\mu); \mu)$ where p symbolizes all the momenta. Considering that the function F_R is homogeneous in the momentum and mass, it may be written, under the scaling transformation of momentum $p = \lambda p_0$, as follows

$$F_R(p; g_R, m_R, \xi_R, \mu) = \lambda^{D_F} F_R(p_0; g_R, \frac{m_R}{\lambda}, \xi_R; \frac{\mu}{\lambda}) \quad (2. 4)$$

where D_F is the canonical dimension of F . Since the renormalization point is a momentum taken to subtract the divergence, we may set $\mu = \mu_0 \lambda$ where $\lambda = e^t$ which is taken to be the same as in $p = p_0 \lambda$. Noticing the above transformation, the solution of the RGE in Eq.(2.2) can be expressed as¹⁵

$$\begin{aligned} F_R(p; g_R, m_R, \xi_R, \mu_0) &= \lambda^{D_F} e^{\int_1^\lambda \frac{d\lambda}{\lambda} \gamma_F(\lambda)} F_R(p_0; \\ &g_R(\lambda), m_R(\lambda) \lambda^{-1}, \xi_R(\lambda); \mu_0) \end{aligned} \quad (2. 5)$$

where $g_R(\lambda)$, $m_R(\lambda)$ and $\xi_R(\lambda)$ are the running coupling constant, the running mass and the running gauge parameter, respectively. The solution written above shows the behavior of the function F_R under the scaling of momenta.

How to determine the function $F_R(p_0; \dots, \mu_0)$ on the RHS of Eq.(2.5) when the $F_R(p_0, \dots)$ stands for a wave function, a propagator or a vertex? This question can be unambiguously answered in MOM scheme, but was not answered clearly in the literature^{27,28}. Noticing that the momentum p_0 and the renormalization point μ_0 are fixed, but may be chosen arbitrarily, we may, certainly, set $p_0^2 = \mu_0^2$. With this choice, by making use of the following boundary condition satisfied by a propagator, a vertex or a wave function

$$F_R(p_0; g_R, m_R, \xi_R, \mu) |_{p_0^2 = \mu_0^2} = F_R^{(0)}(p_0; g_R, m_R, \xi_R) \quad (2. 6)$$

where the function $F_R^{(0)}(p; g_R, m_R, \alpha_R)$ is of the form of free propagator, bare vertex or free wave function and independent of the renormalization point (see the examples given in the next section) and considering the homogeneity of the function F_R as mentioned in Eq.(2.4), we may write

$$\lambda^{D_F} F_R(p_0; g_R(\lambda), m_R(\lambda)\lambda^{-1}, \xi_R(\lambda), \mu_0) |_{p_0^2 = \mu_0^2} = F_R^{(0)}(p; g_R(\lambda), m_R(\lambda), \xi_R(\lambda)) \quad (2. 7)$$

where the renormalized coupling constant, mass and vertex in the function $F_R^{(0)}(p, \dots)$ become the running ones. With the expression given in Eq.(2.7), Eq.(2.5) will finally be written in the form

$$F_R(p; g_R, m_R, \xi_R) = e^{\int_1^\lambda \frac{d\lambda}{\lambda} \gamma_F(\lambda)} F_R^{(0)}(p; g_R(\lambda), m_R(\lambda), \xi_R(\lambda)) \quad (2. 8)$$

For a gauge field theory, it is easy to check that the anomalous dimension in Eq.(2.8) will be cancelled out in S-matrix elements. To show this point more specifically, let us take the two-electron scattering taking place in t-channel as an example. Considering that a S-matrix element expressed in terms of unrenormalized quantities is equal to that represented by the corresponding renormalized quantities, the scattering amplitude may be written as

$$S_{fi} = \bar{u}_R^{\alpha'}(p'_1) \Gamma_R^\mu(p'_1, p_1) u_R^\alpha(p_1) i D_{R\mu\nu}(k) \bar{u}_R^{\beta'}(p'_2) \Gamma_R^\nu(p'_2, p_2) u_R^\beta(p_2) \quad (2. 9)$$

where $k = p'_1 - p_1 = p_2 - p'_2$; $u_R^\alpha(p)$, $\Gamma_R^\mu(p', p)$ and $i D_{R\mu\nu}(k)$ represent the fermion wave function, the proper vertex and the photon propagator respectively which are all renormalized. The renormalization constants of the wave

function, the propagator and the vertex will be designated by $\sqrt{Z_2}, Z_3$ and Z_Γ respectively. The constant Z_Γ is defined as

$$Z_\Gamma = Z_2^{-1} Z_3^{-\frac{1}{2}} \quad (2. 10)$$

because the vertex in Eq.(2.9) contains a coupling constant in it.

According to the formula given in Eq.(2.8), the renormalized fermion wave function, photon propagator and vertex can be represented in the forms as shown below. For the fermion wave function, we have

$$u_R^\alpha(p) = e^{\int_1^\lambda \frac{d\lambda}{\lambda} \gamma_F(\lambda)} u_{R\alpha}^{(0)}(p, m_R(\lambda)) \quad (2. 11)$$

where

$$u_{R\alpha}^{(0)}(p, m_R(\lambda)) = \left(\frac{E + m_R(\lambda)}{2m_R(\lambda)} \right)^{\frac{1}{2}} \left(\frac{\vec{\sigma} \cdot \vec{p}}{E + m_R(\lambda)} \right) \varphi_\alpha(\vec{p}) \quad (2. 12)$$

is the free wave function in which $m_R(\lambda)$ is the running mass and

$$\gamma_F = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_2 \quad (2. 13)$$

is the anomalous dimension of fermion wave function. For the renormalized photon propagator, we can write

$$iD_{R\mu\nu}(k) = e^{\int_1^\lambda \frac{d\lambda}{\lambda} \gamma_3(\lambda)} iD_{R\mu\nu}^{(0)}(k)$$

where

$$iD_{R\mu\nu}^{(0)}(k) = -\frac{i}{k^2 + i\varepsilon} [g_{\mu\nu} - (1 - \alpha_R(\lambda)) \frac{k_\mu k_\nu}{k^2}] \quad (2. 14)$$

is the free propagator with $\alpha_R(\lambda)$ being the running gauge parameter in it and

$$\gamma_3(\lambda) = \mu \frac{d}{d\mu} \ln Z_3 \quad (2. 15)$$

is the anomalous dimension of the propagator. For the renormalized vertex, it reads

$$\Gamma_R^\mu(p', p) = e^{\int_1^\lambda \frac{d\lambda}{\lambda} \gamma_\Gamma(\lambda)} \Gamma_R^{(0)\mu}(p', p) \quad (2. 16)$$

where

$$\Gamma_R^{(0)\mu}(p', p) = ie_R(\lambda)\gamma^\mu \quad (2.17)$$

is the bare vertex containing the running coupling constant (the electric charge) $e_R(\lambda)$ in it and

$$\gamma_\Gamma(\lambda) = \mu \frac{d}{d\mu} \ln Z_\Gamma = -\mu \frac{d}{d\mu} \ln Z_2 - \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_3 \quad (2.18)$$

is the anomalous dimension of the vertex here the relation in Eq.(2.10) has been used. Upon substituting Eqs.(2.11), (2.14) and (2.17) into Eq.(2.9) and noticing Eqs.(2.13), (2.16) and (2.19), we find that the anomalous dimensions in the S-matrix element are all cancelled out with each other. As a result, we arrive at

$$S_{fi} = \bar{u}_{R\alpha'}^{(0)}(p'_1) \Gamma_R^{(0)\mu}(p'_1, p_1) u_{R\alpha}^{(0)}(p_1) i D_{R\mu\nu}^{(0)}(k) \bar{u}_{R\beta'}^{(0)}(p'_2) \Gamma_R^{(0)\nu}(p'_2, p_2) u_{R\beta}^{(0)}(p_2) \quad (2.19)$$

This expression clearly shows that the exact S-matrix element of the two-electron scattering can be represented in the form as given in the lowest order (tree diagram) approximation except that all the physical parameters in the matrix elements are replaced by their effective (running) ones. For other S-matrix elements, the conclusion is the same. This is because any S-matrix element is unexceptionably expressed in terms of a number of wave functions, propagators and proper vertices each of which can be represented in the form as shown in Eq.(2.8) and the anomalous dimensions in the matrix element, as can be easily proved, are all cancelled out eventually. This result and the fact that any S-matrix element is independent of the gauge parameter (This is the so-called gauge-invariance of S-matrix which is implied by the unitarity of S-matrix elements) indicate that the task of renormalization for a gauge field theory is reduced to find the running coupling constant and the running mass by their RGEs. These running quantities completely describe the effect of higher order perturbative corrections.

3. Ward Identity

In QED renormalization, the following Ward identity plays crucial role ⁶

$$\Lambda_\mu(p'; p)|_{p'=p} = -\frac{\partial \Sigma(p)}{\partial p_\mu} \quad (3.1)$$

where $\Lambda_\mu(p', p)$ represents the vertex correction which is defined by taking out a coupling constant e and $\Sigma(p)$ denotes the fermion self-energy. Firstly, we show how the above identity determines the subtraction of the fermion self-energy $\Sigma(p)$. According to the Ward identity, the divergence in $\Lambda_\mu(p', p)$ should be, in the GMS scheme, subtracted at a time-like (Minkowski) renormalization point for the momenta of the external fermion lines, $p^2 = p'^2 = \mu^2$ which implies $\not{p} = \not{p}' = \mu$. When $\mu = m$ (the fermion mass), we will come to the subtraction in the OS scheme. In the case of $\mu \neq m$, the subtraction is defined on a generalized mass shell. At the renormalization point μ , we have

$$\Lambda_\mu(p', p) |_{p'=\not{p}=\mu} = L\gamma_\mu \quad (3. 2)$$

Thus, the vertex correction may be represented as

$$\Lambda_\mu(p', p) = L\gamma_\mu + \Lambda_\mu^c(p', p) \quad (3. 3)$$

where L is a divergent constant depending on μ and $\Lambda_\mu^c(p', p)$ is the finite correction satisfying the boundary condition

$$\Lambda_\mu^c(p', p) |_{p'=\not{p}=\mu} = 0 \quad (3. 4)$$

On inserting Eq.(3.3) into Eq.(3.1) and integrating the both sides of Eq.(3.1) over the momentum p^μ , we get

$$\Sigma(p) - \Sigma(\mu) = -(\not{p} - \mu)L - \int_{p_0^\mu}^{p^\mu} dp^\mu \Lambda_\mu^c(p, p) \quad (3. 5)$$

where the momentum p_0 is chosen to make $\not{p}_0 = \mu$. Since the last term on the RHS of Eq.(3.5) vanishes when $p^\mu \rightarrow p_0^\mu$, we may write

$$\int_{p_0^\mu}^{p^\mu} dp^\mu \Lambda_\mu^c(p, p) = (\not{p} - \mu)C(p^2) \quad (3. 6)$$

where $C(p^2)$ is a convergent function satisfying the following boundary condition

$$C(p^2) |_{p^2=\mu^2} = 0 \quad (3. 7)$$

which is implied by Eq.(3.4). Substituting Eq.(3.6) into Eq.(3.5) and setting

$$\Sigma(\mu) = A \quad (3. 8)$$

and

$$L = -B \quad (3. 9)$$

the self-energy is finally written as

$$\Sigma(p) = A + (\not{p} - \mu)[B - C(p^2)] \quad (3. 10)$$

where the constants A and B have absorbed all the divergences appearing in the $\Sigma(p)$. The above derivation shows that the subtraction given in Eq.(3.10) is uniquely correct in the GMS scheme as it is compatible with the Ward identity. According to the subtraction in Eq.(3.3), the full vertex can be written as

$$\begin{aligned} \Gamma_\mu(p', p) &= \gamma_\mu + \Lambda_\mu(p', p) \\ &= Z_1^{-1} \Gamma_\mu^R(p', p) \end{aligned} \quad (3. 11)$$

where Z_1 is the vertex renormalization constant defined as

$$Z_1^{-1} = 1 + L \quad (3. 12)$$

and $\Gamma_\mu^R(p', p)$ is the renormalized vertex represented by

$$\Gamma_\mu^R(p', p) = \gamma_\mu + \Lambda_\mu^R(p', p) \quad (3. 13)$$

which satisfies the boundary condition

$$\Gamma_\mu^R(p', p) \mid_{p'=\not{p}=\mu} = \gamma_\mu \quad (3. 14)$$

Based on the subtraction given in Eq.(3.10), the full fermion propagator may be renormalized in such a way

$$iS_F(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\varepsilon} = Z_2 iS_F^R(p) \quad (3. 15)$$

where Z_2 is the propagator renormalization constant defined by

$$Z_2^{-1} = 1 - B \quad (3. 16)$$

and $S_F^R(p)$ denotes the renormalized propagator represented as

$$S_F^R(p) = \frac{i}{\not{p} - m_R - \Sigma_R(p) + i\varepsilon} \quad (3. 17)$$

which has a boundary condition as follows

$$S_F^R(p) \mid_{p^2=\mu^2} = \frac{i}{\not{p} - m_R} \quad (3.18)$$

In Eq.(3.17), m_R and $\Sigma_R(p)$ designate the renormalized mass and the finite correction of the self-energy respectively. The renormalized mass is defined by

$$m_R = Z_m^{-1} m \quad (3.19)$$

where Z_m is the mass renormalization constant expressed by

$$Z_m^{-1} = 1 + Z_2 [A m^{-1} + (1 - \mu m^{-1}) B] \quad (3.20)$$

Particularly, from Eqs.(3.9), (3.12) and (3.16). it is clear to see

$$Z_1 = Z_2 \quad (3.21)$$

This just is the Ward identity obeyed by the renormalization constants.

Let us verify whether the Ward identity is fulfilled in the one-loop approximation. The Feynman integrals of one-loop diagrams in QED have been calculated in the literature by various regularization procedures^{3-6,14,15}. In the GMS scheme, the fermion self-energy depicted in Fig.(1a), according to the dimensional regularization procedure, is regularized in the form

$$\begin{aligned} \Sigma(p) = & -\frac{e^2}{(4\pi)^2} (4\pi M^2)^\varepsilon \Gamma(1 + \varepsilon) \int_0^1 dx \left\{ \frac{1}{\varepsilon \Delta(p)^\varepsilon} [2(1 - \varepsilon) \right. \\ & \times (1 - x) \not{p} - (4 - 2\varepsilon)m + (1 - \xi)(m - 2x \not{p})] - 2(1 - \xi) \\ & \times (1 - x) \frac{x^2 p^2 \not{p}}{\Delta(p)} \} \end{aligned} \quad (3.22)$$

where $\varepsilon = 2 - \frac{n}{2}$,

$$\Delta(p) = p^2 x(x - 1) + m^2 x \quad (3.23)$$

and M is an arbitrary mass introduced to make the coupling constant e to be dimensionless in the space of dimension n . According to the definition shown in Eq.(3.8) and noticing

$$p^2 \not{p} = (\not{p} - \mu)[p^2 + \mu(\not{p} + \mu)] + \mu^3 \quad (3.24)$$

one can get from Eq.(3.22)

$$\begin{aligned}
A = & -\frac{e^2}{(4\pi)^2} (4\pi M^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \left\{ \frac{1}{\varepsilon \Delta(\mu)^\varepsilon} \right. \\
& \times [2\mu[1 + (\xi - 2)x - \varepsilon(1-x)] - (3 + \xi - 2\varepsilon)m] \\
& \left. - 2(1-\xi)(1-x)x^2 \frac{\mu^3}{\Delta(\mu)^{1+\varepsilon}} \right\} \quad (3.25)
\end{aligned}$$

where

$$\Delta(\mu) = x[\mu^2(x-1) + m^2] \quad (3.26)$$

On substituting Eqs.(3.22) and (3.25) in Eq.(3.10), it is found that

$$\begin{aligned}
B = & [\Sigma(p) - A](p - \mu)^{-1} |_{p=\mu} \\
= & -\frac{e^2}{(4\pi)^2} (4\pi M^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \left\{ \frac{1}{\varepsilon \Delta(\mu)^\varepsilon} [2(1-\varepsilon) \right. \\
& \times (1-x) - 2(1-\xi)x] + \frac{2\mu^2}{\Delta(\mu)^{1+\varepsilon}} [2(1-\varepsilon)x(x-1)^2 \\
& + 5(1-\xi)x^2(x-1) + \frac{m}{\mu}(3 + \xi - 2\varepsilon)x(x-1)] - \frac{4\mu^4}{\Delta(\mu)^{2+\varepsilon}} \\
& \left. \times (1-\xi)(1+\varepsilon)(x-1)^2 x^3 \right\} \quad (3.27)
\end{aligned}$$

For the diagram of one-loop vertex correction shown in Fig.(1b), according to the definition written in Eq.(3.2), it is not difficult to obtain, in the n-dimensional space, the regularized form of the constant L

$$\begin{aligned}
L = & \frac{e^2}{(4\pi)^2} (4\pi M^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \left\{ \frac{2x}{\varepsilon \Delta(\mu)^\varepsilon} [\varepsilon(\varepsilon - \frac{3}{2}) \right. \\
& + \xi(1 - \frac{1}{2}\varepsilon)] - \frac{x}{\Delta(\mu)^{1+\varepsilon}} [2\mu^2(\varepsilon - 1)(x-1)^2 \\
& - (1-\xi)(x^2 - x - 1)\mu^2 + \varepsilon(1-\xi)(x-1)\mu^2 - 4m\mu[(2-\varepsilon) \\
& \times (x-1) + \frac{1}{2}(1-\xi)(2-3x) + \frac{1}{2}\varepsilon(1-\xi)(x-1)] + m^2[2(\varepsilon \\
& - 1) - (1-\varepsilon)(1-\xi)(x-1)]] + (1-\xi) \frac{(1+\varepsilon)}{\Delta(\mu)^{2+\varepsilon}} (x-1)x^3\mu^2 \\
& \left. \times (m + \mu)^2 \right\} \quad (3.28)
\end{aligned}$$

With the expressions given in Eqs.(3.27) and (3.28), in the approximation of order e^2 , the renormalization constants defined in Eq.(3.12) and (3.16) will be represented as $Z_1 = 1 - L$ and $Z_2 = 1 + B$. In the limit $\varepsilon \rightarrow 0$, these constants are divergent, being not well-defined. So, to verify the Ward identity in Eq.(3.21), it is suitable to see whether their anomalous dimensions satisfy the corresponding identity

$$\gamma_1 = \gamma_2 \quad (3. 29)$$

where $\gamma_i(\mu) = \lim_{\varepsilon \rightarrow 0} \mu \frac{d}{d\mu} \ln Z_i(\mu, \varepsilon)$ ($i=1,2$). For the renormalization group calculations, in practice, the above identity is only necessary to be required. Through direct calculation by using the constants in Eqs.(3.27) and (3.28), it is easy to prove

$$\begin{aligned} \gamma_1 &= \gamma_2 \\ &= -\frac{e^2}{(4\pi)^2} \{6\xi - 6(3 + \xi)\sigma + 12\xi\sigma^2 + 6(3 + \xi - 2\xi\sigma) \\ &\quad \times \sigma^3 \ln \frac{\sigma^2}{\sigma^2 - 1} + 4[2\xi - (3 + \xi)\sigma] \frac{1}{\sigma^2 - 1}\} \end{aligned} \quad (3. 30)$$

where $\sigma = \frac{m}{\mu}$. This identity guarantees the correctness of the one-loop renormalizations in the GMS scheme. In the zero-mass limit ($\sigma \rightarrow 0$), the identity in Eq.(3.30) reduces to the result given in the MS scheme

$$\gamma_1 = \gamma_2 = \frac{\xi e^2}{8\pi^2} \quad (3. 31)$$

This result can directly be derived from such expressions of the constants B and L that they are obtained from Eqs.(3.27) and (3.28) by setting $m=0$. It is easy to see that in these expressions, only the terms proportional ε^{-1} give nonvanishing contributions to the anomalous dimensions. However, in the case of $m \neq 0$, the terms proportional to ε^{-1} in Eqs.(3.27) and (3.28) give different results. In this case, to ensure the identity in Eq.(3.29) to be satisfied, the other terms without containing ε^{-1} in Eqs.(3.27) and (3.28) must be taken into account.

4. Effective Coupling Constant

The RGE for the renormalized coupling constant may be immediately written out from Eq.(2.2) by setting $F = e$,

$$\mu \frac{d}{d\mu} e_R(\mu) + \gamma_e(\mu) e_R(\mu) = 0 \quad (4. 1)$$

In the above, the anomalous dimension $\gamma_e(\mu)$ as defined in Eq.(2.3) is now determined by the following renormalization constant⁵

$$Z_e = \frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}} = Z_3^{-\frac{1}{2}} \quad (4. 2)$$

where the identity in Eq.(3.21) has been considered. The photon propagator renormalization constant Z_3 is, in the GMS scheme, defined by

$$Z_3^{-1} = 1 + \Pi(\mu^2) \quad (4. 3)$$

where $\Pi(\mu^2)$ is the scalar function appearing in the photon self-energy tensor $\Pi_{\mu\nu} = (k_\mu k_\nu - k^2 g_{\mu\nu})\Pi(\mu^2)$. In view of Eq.(4.2), we can write

$$\gamma_e = \lim_{\varepsilon \rightarrow 0} \mu \frac{d}{d\mu} \ln Z_e = -\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \mu \frac{d}{d\mu} \ln Z_3 \quad (4. 4)$$

For the one-loop diagram represented in Fig.(1c), according to Eq.(4.3), it is easy to derive the regularized form of the constant Z_3 by the dimensional regularization procedure

$$Z_3 = 1 + \frac{e^2}{4\pi^2} (4\pi M^2)^\varepsilon (2 - \varepsilon) \frac{\Gamma(1 + \varepsilon)}{\varepsilon} \int_0^1 \frac{dx x(x-1)}{[\mu^2 x(x-1) + m^2]^\varepsilon} \quad (4. 5)$$

Substituting Eq.(4.5) into Eq.(4.4), it is found that

$$\gamma_e = -\frac{e^2}{12(\pi)^2} \left\{ 1 + 6\sigma^2 + \frac{12\sigma^4}{\sqrt{1-4\sigma^2}} \ln \frac{1 + \sqrt{1-4\sigma^2}}{1 - \sqrt{1-4\sigma^2}} \right\} \quad (4. 6)$$

where $\sigma = \frac{m}{\mu}$. In this expression, the charge e and the mass m are unrenormalized. In the approximation of order e^2 , they can be replaced by the

renormalized ones e_R and m_R because in this approximation, as pointed out in the previous literature¹⁵, the lowest order approximation of the relation between the $e(m)$ and the $e_R(m_R)$ is only necessary to be taken into account. Furthermore, when we introduce the scaling variable λ for the renormalization point and set $\mu_0 = m_R$ (which can always be done since the μ_0 is fixed, but may be chosen at will), we have $\sigma = \frac{m_R}{\mu_0 \lambda} = \frac{1}{\lambda}$. Thus, with the expressions of Eq.(4.6), Eq.(4.1) may be rewritten in the form

$$\lambda \frac{de_R(\lambda)}{d\lambda} = \beta(\lambda) \quad (4. 7)$$

where

$$\begin{aligned} \beta(\lambda) &= -\gamma_e(\lambda)e_R(\lambda) \\ &= \frac{e_R^3(\lambda)}{12\pi^2}F_e(\lambda) \end{aligned} \quad (4. 8)$$

in which

$$F_e(\lambda) = 1 + \frac{6}{\lambda^2} + \frac{12}{\lambda^4}f(\lambda) \quad (4. 9)$$

$$\begin{aligned} f(\lambda) &= \frac{\lambda}{\sqrt{\lambda^2 - 4}} \ln \frac{\lambda + \sqrt{\lambda^2 - 4}}{\lambda - \sqrt{\lambda^2 - 4}} \\ &= \begin{cases} \frac{2\lambda}{\sqrt{4-\lambda^2}} \cot^{-1} \frac{\lambda}{\sqrt{4-\lambda^2}}, & \text{if } \lambda \leq 2 \\ \frac{2\lambda}{\sqrt{\lambda^2-4}} \coth^{-1} \frac{\lambda}{\sqrt{\lambda^2-4}}, & \text{if } \lambda \geq 2 \end{cases} \end{aligned} \quad (4. 10)$$

Upon substituting Eqs.(4.8)-(4.10) into Eq.(4.7) and then integrating Eq.(4.7) by applying the familiar integration formulas, the effective (running) coupling constant will be found to be

$$\alpha_R(\lambda) = \frac{\alpha_R}{1 - \frac{2\alpha_R}{3\pi}G(\lambda)} \quad (4. 11)$$

where $\alpha_R(\lambda) = \frac{e_R^2(\lambda)}{4\pi}$, $\alpha_R = \alpha_R(1)$ and

$$\begin{aligned} G(\lambda) &= \int_1^\lambda \frac{d\lambda}{\lambda} F_e(\lambda) \\ &= 2 + \sqrt{3}\pi - \frac{2}{\lambda^2} + \left(1 + \frac{2}{\lambda^2}\right) \frac{1}{\lambda} \varphi(\lambda) \end{aligned} \quad (4. 12)$$

in which

$$\begin{aligned}\varphi(\lambda) &= \sqrt{\lambda^2 - 4} \ln \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) \\ &= \begin{cases} -\sqrt{4 - \lambda^2} \cos^{-1} \frac{\lambda}{2}, & \text{if } \lambda \leq 2 \\ \sqrt{\lambda^2 - 4} \cosh^{-1} \frac{\lambda}{2}, & \text{if } \lambda \geq 2 \end{cases} \quad (4.13)\end{aligned}$$

As mentioned in Sect.2, the variable λ is also the scaling parameter of momenta, $p = \lambda p_0$ and it is convenient to put $p_0^2 = \mu_0^2$ so as to apply the boundary condition. Thus, owing to the choice $\mu_0 = m_R$, we have $p_0^2 = m_R^2$ and $\lambda = (\frac{p^2}{m_R^2})^{\frac{1}{2}}$. In this case, it is apparent that when $\lambda = 1$, Eq.(4.11) will be reduced to the result given on the mass shell, $\alpha_R(1) = \alpha_R = \frac{1}{137}$ which is identified with that as measured in experiment.

The behavior of the $\alpha_R(\lambda)$ are exhibited in Figs.(2) and (3). For small λ , Eq.(4.11) may be approximated by

$$\alpha_R(\lambda) \approx \frac{3}{4} \lambda^3 \quad (4.14)$$

It is clear that when $\lambda \rightarrow 0$, the $\alpha_R(\lambda)$ tends to zero. This desirable behavior, which indicates that at large distance (small momentum) the interacting particles decouple, is completely consistent with our knowledge about the electromagnetic interaction. For large momentum (small distance), Eq.(4.11) will be approximated by

$$\alpha_R(\lambda) \approx \frac{\alpha_R}{1 - \frac{2\alpha_R}{3\pi} \ln \lambda} \quad (4.15)$$

This result was given previously in the mass-independent MS scheme. In the latter scheme, the β -function in Eq.(4.8) is only a function of the $e_R(\lambda)$ since $F_e(\lambda) = 1$ due to $m = 0$ in this case. But, in general, the mass of a charged particle is not zero. Therefore, the result given in the MS scheme can only be viewed as an approximation in the large momentum limit from the viewpoint of conventional perturbation theory. Fig.(3) shows that the $\alpha_R(\lambda)$ increases with the growth of λ and tends to infinity when the λ approaches an extremely large value $\lambda_0 \approx e^{\frac{2\pi}{3\alpha_R}} = e^{287}$ (the Landau pole). If the λ goes from λ_0 to infinity, we find, the $\alpha_R(\lambda)$ will become negative and tends to zero. This result is unreasonable, conflicting with the physics. The unreasonableness indicates that in the region $[\lambda_0, \infty)$, the QED perturbation theory and even the QED itself is invalid⁵.

5. Effective Fermion Mass

The RGE for a renormalized fermion mass can be directly read from Eq.(2.2) when we set $F = m_R$. Noticing $\mu \frac{d}{d\mu} = \lambda \frac{d}{d\lambda}$, this equation may be written in the form

$$\lambda \frac{dm_R(\lambda)}{d\lambda} = -\gamma_m(\lambda)m_R(\lambda) \quad (5. 1)$$

where the anomalous dimension $\gamma_m(\lambda)$, according to the definition in Eq.(2.3), can be derived from the renormalization constant represented in Eq.(3.20). At one-loop level, by making use of the constants A , B and Z_2 which were written in Eqs.(3.25), (3.27) and (3.16) respectively, in the approximation of order e^2 , it is not difficult to derive

$$\gamma_m(\lambda) = \lim_{\varepsilon \rightarrow 0} \mu \frac{d}{d\mu} \ln Z_m = \frac{e_R^2}{(4\pi)^2} F_m(\lambda) \quad (5. 2)$$

where

$$\begin{aligned} F_m(\lambda) = & 2\xi\lambda + 6[3 + 2\xi - \frac{3(1 + \xi)}{\lambda} + \frac{2\xi}{\lambda^2}] \\ & - \frac{12(1 + \xi)\lambda}{1 + \lambda} + 6[3 + \xi - \frac{3(1 + \xi)}{\lambda} + \frac{2\xi}{\lambda^2}] \frac{1}{\lambda^2} \ln |1 - \lambda^2| \end{aligned} \quad (5. 3)$$

here the relation $\sigma = \frac{1}{\lambda}$ has been used. Inserting Eq.(5.2) into Eq.(5.1) and integrating the latter equation, one may obtain

$$m_R(\lambda) = m_R e^{-S(\lambda)} \quad (5. 4)$$

This just is the effective (running) fermion mass where $m_R = m_R(1)$ which is given on the mass-shell and

$$S(\lambda) = \frac{1}{4\pi} \int_1^\lambda \frac{d\lambda}{\lambda} \alpha_R(\lambda) F_m(\lambda) \quad (5. 5)$$

In the above, the bare charge appearing in Eq.(5.2) has been replaced by the renormalized one and further by the running one shown in Eq.(4.11). If the coupling constant in Eq.(5.5) is taken to be the constant defined on the mass-shell, the integral over λ can be explicitly calculated. The result is

$$S(\lambda) = \frac{\alpha_R}{4\pi} [\varphi_1(\lambda) + \xi \varphi_2(\lambda)] \quad (5. 6)$$

where

$$\varphi_1(\lambda) = 3(1-\lambda)\left\{\frac{2}{\lambda} + \left[\frac{2}{\lambda^3} - \frac{1}{\lambda^2}(1+\lambda)\right]\ln|1-\lambda^2|\right\} \quad (5.7)$$

and

$$\varphi_2(\lambda) = (1-\lambda)\left\{(2\lambda-3)\frac{1}{\lambda^4}\ln|1-\lambda^2| - \frac{3(1-\lambda)}{\lambda^2}\right\} - 2 \quad (5.8)$$

in which

$$\ln|1-\lambda^2| = \begin{cases} 2[\ln(1+\lambda) - \tanh^{-1}\lambda], & \text{if } \lambda \leq 1 \\ 2[\ln(1+\lambda) - \coth^{-1}\lambda], & \text{if } \lambda \geq 1 \end{cases} \quad (5.9)$$

As we see, the function $S(\lambda)$ and hence the effective mass $m_R(\lambda)$ are gauge-dependent. The gauge-dependence is displayed in Fig.(4). The figure shows that for $\xi < 10$, the effective masses given in different gauges are almost the same and behave as a constant in the region of $\lambda < 1$, while, in the region of $\lambda > 1$, they all tend to zero with the growth of λ . But, the $m_R(\lambda)$ given in the gauge of $\xi \neq 0$ goes to zero more rapidly than the one given in the Landau gauge ($\xi = 0$). To be specific, in the following we show the result given in the Landau gauge which was regarded as the preferred gauge in the literature^{3,29},

$$m_R(\lambda) = m_R e^{-\frac{\alpha_R}{4\pi}\varphi_1(\lambda)} \quad (5.10)$$

In the limit $\lambda \rightarrow 0$,

$$m_R(\lambda) \rightarrow m_R e^{-\frac{3\alpha_R}{4\pi}} = 1.001744m_R \quad (5.11)$$

This clearly indicates that when λ varies from 1 to zero, the $m_R(\lambda)$ almost keeps unchanged. This result physically is reasonable. Whereas, in the region of $\lambda > 1$, the $m_R(\lambda)$ decreases with increase of λ and goes to zero near the critical point λ_0 . This behavior suggests that at very high energy, the fermion mass may be ignored in the evaluation of S-matrix elements.

6. Comments and Discussions

In this paper, the QED renormalization has been restudied in the GMS scheme. The exact and explicit expressions of the one-loop effective coupling

constant and fermion mass are obtained in the mass-dependent renormalization scheme and show reasonable asymptotic behaviors. A key point to achieve these results is that the subtraction is performed in the way of respecting the Ward identity, i.e. the gauge symmetry. For comparison, it is mentioned that in some previous literature^{15,19}, the fermion self-energy is represented in such a form

$$\Sigma(p) = A(p^2) \not{p} + B(p^2)m \quad (6. 1)$$

If we subtract the divergence in the $\Sigma(p)$ at the renormalization point $p^2 = \mu^2$, the fermion propagator is still expressed in the form as written in Eqs.(3.15) and (3.17); but, the renormalization constants Z_2 and Z_m are now defined by

$$Z_2^{-1} = 1 - A(\mu^2), Z_m^{-1} = Z_2[1 + B(\mu^2)] \quad (6. 2)$$

The one-loop expressions of the $A(\mu^2)$ and $B(\mu^2)$ can directly be read from Eq.(3.22). Here, we show the one-loop anomalous dimension of fermion propagator which is given by the above subtraction

$$\begin{aligned} \gamma_2 &= \lim_{\varepsilon \rightarrow 0} \mu \frac{d}{d\mu} \ln Z_2 \\ &= \frac{\xi e^2}{4\pi^2} \left[\frac{1}{2} + \sigma^2 - \sigma^4 \ln \frac{\sigma^2}{\sigma^2 - 1} \right] \end{aligned} \quad (6. 3)$$

In comparison of the above γ_2 with the γ_1 shown in Eq.(3.30), we see, the Ward identity in Eq.(3.29) can not be fulfilled unless in the zero-mass limit ($\sigma \rightarrow 0$). Since the Ward identity is an essential criterion to identify whether a subtraction is correct or not, the subtraction stated above should be excluded from the mass-dependent renormalization.

Another point we would like to address is that in the Ward identity shown in Eq.(3.1), the momenta p and p' on the fermion lines in the vertex are set to be equal. According to the energy-momentum conservation, the momentum k on the photon line should be equal to zero. correspondingly, the subtraction shown in Eq.(3.2) was carried out at the so-called asymmetric points, $p'^2 = p^2 = \mu^2$ and $k^2 = (p' - p)^2 = 0$. These subtraction points coincide with the energy- momentum conservation, i.e., the Lorentz-invariance. Nevertheless, the subtraction performed at the symmetric point

$p'^2 = p^2 = k^2 = -\mu^2$ was often used in the previous works ^{3,19,20}. This subtraction not only makes the calculation too complicated, but also violates the energy-momentum conservation which holds in the vertex. That is why the symmetric point subtraction is beyond our choice.

As mentioned in Introduction, the GMS subtraction is a kind of MOM scheme in which the renormalization points are chosen to be time-like. In contrast, in the conventional MOM scheme^{3,6,29}, the renormalization points were chosen to be space-like, i.e. $p_i^2 = -\mu^2$ which implies $\not{p}_i = i\mu$. In this scheme, the one-loop result for the anomalous dimension γ_e can be written out from Eq.(4.6) by the transformation $\sigma \rightarrow -i\sigma$. That is

$$\gamma_e = -\frac{4e^2}{3(4\pi)^2} \left\{ 1 - 6\sigma^2 + \frac{12\sigma^4}{\sqrt{1+4\sigma^2}} \ln \frac{\sqrt{1+4\sigma^2} + 1}{\sqrt{1+4\sigma^2} - 1} \right\} \quad (6.4)$$

which is identical to that given in Refs.(6) and (29). In comparison of Eq.(6.4) with Eq.(4.6), we see, the coefficients of σ^2 in Eq.(6.4) changes a minus sign. Substituting Eq.(6.4) into Eq.(4.1) and solving the latter equation, we obtain the running coupling constant which is still represented in Eq.(4.11), but the function $G(\lambda)$ in Eq.(4.11) is now given by

$$\begin{aligned} G(\lambda) = & \frac{2}{\lambda^2} - 2 - \frac{\sqrt{\lambda^2 + 4}}{\lambda} \left(\frac{2}{\lambda^2} - 1 \right) \ln \frac{1}{2} (\lambda + \sqrt{\lambda^2 + 4}) \\ & + \sqrt{5} \ln \frac{1}{2} (1 + \sqrt{5}) \end{aligned} \quad (6.5)$$

This expression is obviously different from the corresponding one written in Eqs.(4.12) and (4.13). At small distance ($\lambda \rightarrow \infty$), Eq.(6.5) still gives the approximate expression presented in Eq.(4.15). However, at large distance, as shown in Fig.(2), the $\alpha_R(\lambda)$ behaves almost a constant. When $\lambda \rightarrow 0$, it approaches to a value equal to $0.99986\alpha_R$, unlike the $\alpha_R(\lambda)$ given in Eqs.(4.11)-(4.13) which tends to zero. Let us examine the effective mass. As we have seen from Sect.5, the one-loop effective fermion mass given in the GMS scheme is real. However, in the space-like momentum subtraction, due to $\not{p} = i\mu$, the effective mass will contain an imaginary part. This result can be seen from the function $F_m(\lambda)$ whose one-loop expression given in the usual MOM scheme can be obtained from Eq.(5.3) by the transformation $\lambda \rightarrow i\lambda$ and therefore becomes complex. The both of subtractions may presumably be suitable for different processes of different physical natures.

But, if the effective mass is required to be real, the subtraction at space-like renormalization point should also be ruled out.

As pointed out in Sect.4, the effective coupling constant shown in Eq.(4.15) which was obtained in the MS scheme is only an approximation given in the large momentum limit from the viewpoint of conventional perturbation theory. Why say so? As is well-known, the MS scheme is a mass-independent renormalization scheme in which the fermion mass is set to vanish in the process of subtraction. The reasonability of this scheme was argued as follows^{1,15}. The fermion propagator can be expanded as a series

$$\frac{1}{\not{p} - m} = \frac{1}{\not{p}} + \frac{1}{\not{p}} m \frac{1}{\not{p}} + \frac{1}{\not{p}} m \frac{1}{\not{p}} m \frac{1}{\not{p}} + \dots \quad (6.6)$$

According to this expansion, the massive propagator $\frac{1}{\not{p} - m}$ may be replaced by the massless one $\frac{1}{\not{p}}$. At the same time, the fermion mass, as the coupling constant, can also be treated as an expansion parameter for the perturbation series. Nevertheless, in the mass-dependent renormalization as shown in this paper, the massive fermion propagator is employed in the calculation and only the coupling constant is taken to be the expansion parameter of the perturbation series. Thus, in order to get the perturbative result of a given order of the coupling constant in the mass-dependent renormalization, according to Eq.(6.6), one has to compute an infinite number of terms in the MS scheme. If only the first term in Eq.(6.6) is considered in the MS scheme, the result derived in the this scheme, comparing to the corresponding one obtained in the mass-dependent renormalization, can only be viewed as an approximation given in the large momentum limit. Even if in this limit, a good renormalization scheme should still be required to eliminate the ambiguity and give an unique result. To this end, we may ask whether there should exist the difference between the MS scheme and the \overline{MS} scheme²? As one knows, when the dimensional regularization is employed in the mass-independent renormalization, the MS scheme only subtracts the divergent term having the ε -pole in a Feynman integral which is given in the limit $\varepsilon \rightarrow 0$ and uses this term to define the renormalization constant. While, the \overline{MS} scheme is designed to include the unphysical terms $\gamma - \ln 4\pi$ (here γ is the Euler constant) in the definition of the renormalization constant. The unphysical terms arise from a special analytical continuation of the space-time dimension from n to 4. When the two different renormalization constants mentioned above are inserted into the relation $e = Z_3^{-\frac{1}{2}} e_R$, one would derive

a relation between the two different renormalized coupling constants given in the MS and \overline{MS} schemes if the higher order terms containing the ε -pole are ignored. It would be pointed out that the above procedure of leading to the difference between the MS and \overline{MS} schemes is not appropriate because the procedure is based on direct usage of the divergent form of the renormalization constants. As emphasized in the Introduction, according to the convergence principle, it is permissible to use such renormalization constants to do a meaningful calculation. The correct procedure of deriving a renormalized quantity is to solve its RGE whose solution is uniquely determined by the anomalous dimension (other than the renormalization constant itself) and boundary condition. In computing the anomalous dimension, the rigorous procedure is to start from the regularized form of the renormalization constant. In the regularized form, it is unnecessary and even impossible to divide a renormalization constant into a divergent part and a convergent part. Since the anomalous dimension is a convergent function of ε due to that the factor $\frac{1}{\varepsilon}$ disappears in it, the limit $\varepsilon \rightarrow 0$ taken after the differentiation with respect to the renormalization point would give an unambiguous result. Especially, the unphysical factor $(4\pi)^\varepsilon \Gamma(1+\varepsilon)$ appearing in Eqs.(3.25), (3.27) and (4.5) straightforwardly approaches 1 in the limit. Therefore, the unphysical terms $\gamma - \ln 4\pi$ could not enter the anomalous dimension and the effective coupling constant. Even if we work in the zero-mass limit or in the large momentum regime, we have an only way to obtain the effective coupling constant as shown in Eq.(4.15) in the one-loop approximation. That is to say, it is impossible, in this case, to result in the difference between the MS and \overline{MS} schemes and also the difference between the MOM and the MS schemes.

The above discussions suggest that the ambiguity arising from different renormalization prescriptions may be eliminated by the necessary physical and mathematical requirements as well as the boundary conditions. It is expected that the illustration given in this paper for the QED one-loop renormalization performed in a mass-dependent scheme would provide a clue on how to do the QED multi-loop renormalization and how to give an improved result for the QCD renormalization.

Acknowledgment

The authors wish to thank professor Shi-Shu Wu for useful discussions. This project was supposed in part by National Natural Science Foundation.

Appendix: Illustration of the regularization procedure by a couple of mathematical examples

We believe that if a quantum field theory is built up on the faithful basis of physical principles and really describes the physics, a S-matrix element computed from such a theory is definite to be convergent even though there occur divergences in the perturbation series of the matrix element. The occurrence of divergences in the perturbation series, in general, is not to be a serious problem in mathematics. But, to compute such a series, it is necessary to employ an appropriate regularization procedure. For example, for the following convergent integral

$$f(a) = \int_0^\infty dx e^{-ax} \quad (\text{A. 1})$$

which equals to $\frac{1}{a}$, if we evaluate it by utilizing the series expansion of the exponential function

$$e^{-ax} = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} x^n \quad (\text{A. 2})$$

as we see, the integral of every term in the series is divergent. In this case, interchange of the integration with the summation actually is not permissible. When every integral in the series is regularized, the interchange is permitted and all integrals in the series become calculable. Thus, the correct procedure of evaluating the integral by using the series expansion is as follows

$$\begin{aligned} f(a) &= \lim_{\Lambda \rightarrow \infty} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \int_0^\Lambda dx x^n \\ &= \lim_{\Lambda \rightarrow \infty} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \frac{\Lambda^{n+1}}{n+1} = \lim_{\Lambda \rightarrow \infty} \frac{1}{a} (1 - e^{-a\Lambda}) \\ &= \frac{1}{a} \end{aligned} \quad (\text{A. 3})$$

This example is somewhat analogous to the perturbation series in the quantum field theory and suggests how to do the calculation of the series with the help of a regularization procedure. Unfortunately, in practice, we are not able to compute all the terms in the perturbation series. In this situation, we

can only expect to get desired physical results from a finite order perturbative calculation. How to do it? To show the procedure of such a calculation, let us look at another mathematical example. The following integral

$$F(a) = \int_0^\infty dx \frac{e^{-(x+a)}}{(x+a)^2} \quad (\text{A. 4})$$

where $a > 0$ is obviously convergent. When the exponential function is expanded as the Taylor series, the integral will be expressed as

$$\begin{aligned} F(a) &= \int_0^\infty dx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x+a)^{n-2} = \int_0^\infty \frac{dx}{(x+a)^2} - \int_0^\infty \frac{dx}{x+a} \\ &\quad + \int_0^\infty dx \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} (x+a)^{n-2} \end{aligned} \quad (\text{A. 5})$$

Clearly, in the above expansion, the first term is convergent, similar to the tree-approximate term in the perturbation theory of a quantum field theory, the second term is logarithmically divergent, analogous to the one-loop-approximate term and the other terms amount to the higher order corrections in the perturbation theory which are all divergent. To calculate the integral in Eq.(A.4), it is convenient at first to evaluate its derivative,

$$\frac{dF(a)}{da} = - \int_0^\infty dx \frac{(x+a+2)}{(x+a)^3} e^{-(x+a)} \quad (\text{A. 6})$$

which is equal to $-e^{-a}/a^2$ as is easily seen from integrating it over x by part. In order to get this result from the series expansion, we have to employ a regularization procedure. Let us define

$$F_\Lambda(a) = \int_0^\Lambda \frac{e^{-(x+a)}}{(x+a)^2} \quad (\text{A. 7})$$

Then, corresponding to Eq.(A.5), we can write

$$\begin{aligned} \frac{dF_\Lambda(a)}{da} &= -2 \int_0^\Lambda \frac{dx}{(x+a)^3} + \int_0^\Lambda \frac{dx}{(x+a)^2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} (n-2) \int_0^\Lambda dx (x+a)^{n-3} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [(\Lambda+a)^{n-2} - a^{n-2}] = \frac{e^{-(\Lambda+a)}}{(\Lambda+a)^2} - \frac{e^{-a}}{a^2} \end{aligned} \quad (\text{A. 8})$$

From the above result, it follows that

$$\frac{dF(a)}{da} = \lim_{\Lambda \rightarrow \infty} \frac{dF_\Lambda(a)}{da} = -\frac{e^{-a}}{a^2} \quad (\text{A. 9})$$

This is the differential equation satisfied by the function $F(a)$. Its solution can be expressed as

$$F(a) = F(a_0) - \int_{a_0}^a da \frac{e^{-a}}{a^2} \quad (\text{A. 10})$$

where $a_0 > 0$ is a fixed number which should be determined by the boundary condition of the equation (A.9). Now, let us focus our attention on the second integral in the first equality of Eq.(A.8). This integral is convergent in the limit $\Lambda \rightarrow \infty$ and can be written as

$$\frac{dF_1(a)}{da} = \lim_{\Lambda \rightarrow \infty} \int_0^\Lambda dx \frac{1}{(x+a)^2} = \int_0^\infty dx \frac{1}{(x+a)^2} = \frac{1}{a} \quad (\text{A. 11})$$

Integrating the above equation over a , we get

$$F_1(a) = F_1(a_0) + \ln \frac{a}{a_0} \quad (\text{A. 12})$$

If setting

$$F_1(a_0) = \ln \frac{a_0}{\mu} \quad (\text{A. 13})$$

where μ is a finite number, Eq.(A.12) becomes

$$F_1(a) = \ln \frac{a}{\mu} \quad (\text{A. 14})$$

This result is finite as long as the parameter μ is not taken to be zero and can be regarded as the contribution of the divergent integral $F_1(a)$ appearing in the second term of Eq.(A.5) to the convergent integral $F(a)$. The procedure described above for evaluating the function $F(a)$ much resembles the renormalization group method and the approach proposed in Ref.(9). It shows us how to calculate a finite quantity from its series expansion which contains divergent integrals.

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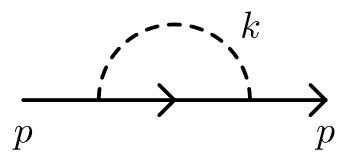
A.1 FIGURE CAPTIONS

Fig.(1) The one-loop diagrams.

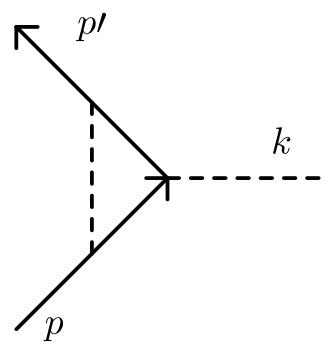
Fig.(2) The one-loop effective coupling constants given in the region of small momenta. The solid curve represents the result obtained at time-like subtraction point. The dashed curve represents the one given at space-like subtraction point.

Fig.(3) The one-loop effective coupling constants for large momenta. The both curves represent the same ones as in Fig.(2).

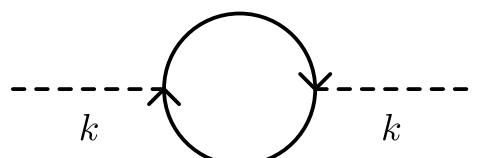
Fig.(4) The one-loop effective electron masses given in the Landau gauge and some other gauges.



(a)



(b)



(c)

Fig.(1)

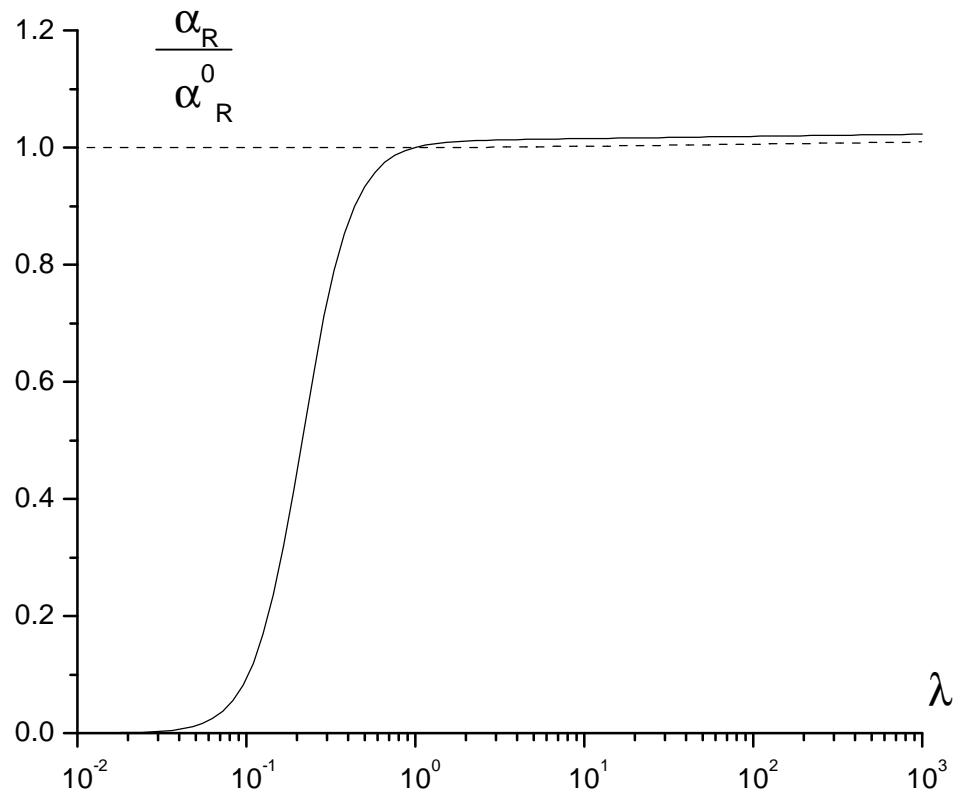


Fig. 2

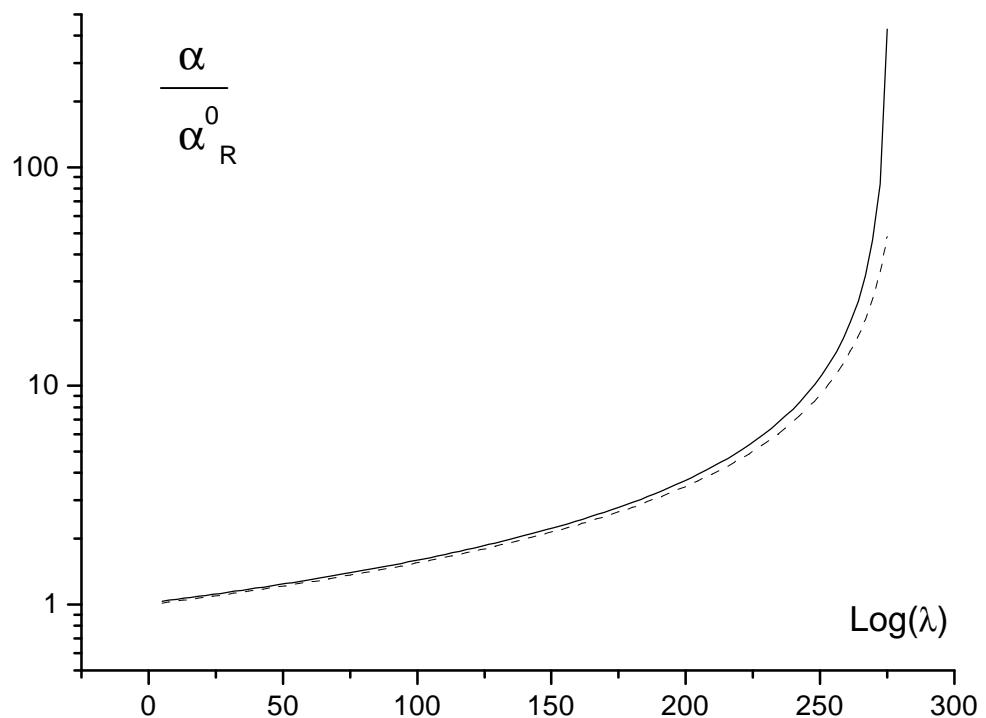


Fig. 3

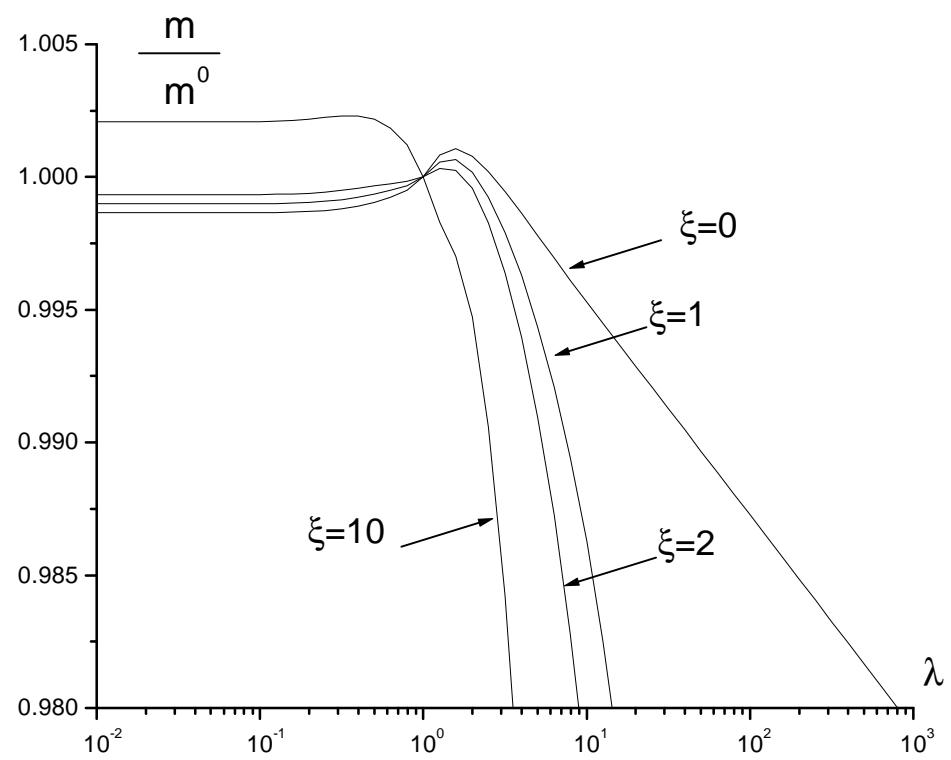


Fig.4